Of the Usage of the Sliding Rule

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1 Introduction

From the nineteenth century up to the 70's, slide rules have been the precious auxiliaries of engineers, architects and technicians worldwide.

Since then, they have been outdated by electronic pocket calculators. Slide rules allow one to quickly perform a lot of numerical computations, such as multiplying, dividing, computing squares, cubes, extracting square and cube roots, trigonometry, etc.

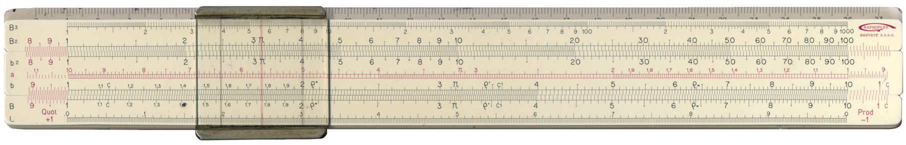

Figure 1: *Graphoplex* slide rule, "Polyphase" type.

In this article, we are going to study in details the principles of a classical slide rule ("Mannheim" or "Polyphase" and derivatives), through the analysis of the underlying mathematical relations and several numerical examples. An high-school level in mathematics should be sufficient enough to read this article.

2 Mathematical principles

All sliding rules are based upon the usage of logarithms. The logarithm is a quite handy mathematical function which allows us to transform a multiplication into an addition:

$$\log(a \times b) = \log(a) + \log(b)$$

The logarithm of the *product* is the *sum* of the logarithms. That way we transform a multiplication, which is quite complex to perform, into an addition, which is much easier. The adding of two values is easy to achieve by the juxtaposition of two physical lengths: to multiply a and b, we add their logarithms. The value c which logarithm is equal to this sum is then equal to the product of a by b.

Following the same idea, we use the equation below to divide two numbers:

$$\log(a/b) = \log(a) - \log(b)$$

A division is tranformed into a substraction.

The same principle is used for manual computation, but by using numerical logarithmic tables.

3 Multiplying

3.1 Principle

Below is a logarithmic scale, on which each graduation a is located at a length of $l_a = \log(a)$ from the origin 1 (figure 2).

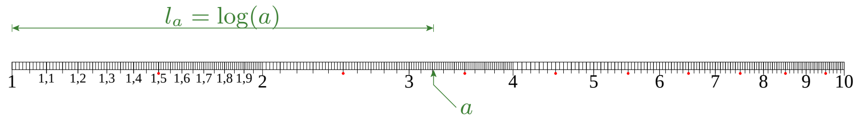

Figure 2: Logarithmic scale.

This logarithmic scale has the noteworthy property that if you translate a cursor on this scale by some length k starting from any number a, you will find this number multiplied by k' ($k = \log(k')$) (figure 3).

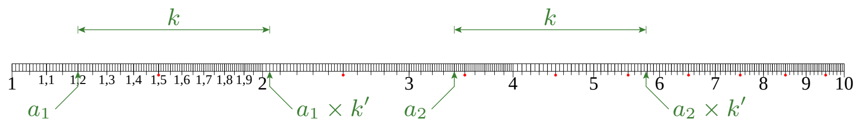

Figure 3: Adding a distance to multiply.

In order to multiply two numbers a and b, we use two logarithmic scales C and D. The length l_a on D between the graduation 1 and the graduation corresponding to the number a is $l_a = \log(a)$. In the same way, the graduation of b on the C scale will give us the length $l_b = \log(b)$. Thus the length $l_c = l_a + l_b$ is the logarithm of $c = a \times b$.

More precisely, we place the two scales C and D side by side, and by translating C of a length l_a we align the base of scale C (the 1) with the graduation of a on scale D. Then, we report length l_b after l_a (starting from scale C base) up to the b graduation on scale C. The length l_c from the base of scale D to this graduation, is $l_c = l_a + l_b$. We can then directly read on scale D the product $a \times b$ (figure 4).

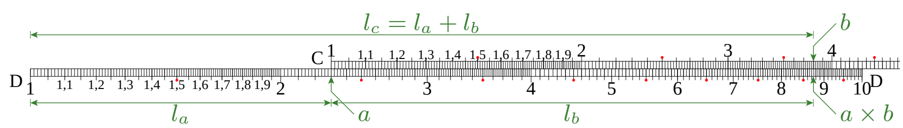

Figure 4: Multiplying a by b.

3.2 C and D?

Nowadays, slide rules are following the first model setup by Amédée Mannheim, a French artillery officer under Napoléon, who labelled the four scales of his rule by the first four letters of the alphabet: A, B, C and D (figure 5). The two logarithmic scales, being situated below, are thus using the C and D letters. Scales B and C slides on a rule relatively to A and D. (We will further see the usage of scales A and B.) The custom has been kept to label the two main logarithmic scales C and D, even if sometimes other symbols are used (notably in France!)

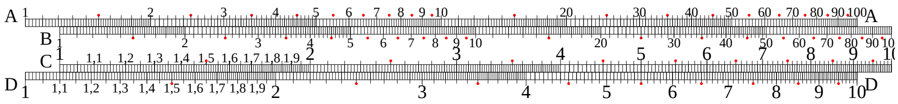

Figure 5: Location of A, B, C and D scales.

3.3 Example

For instance, we will multiply 3.1 by 1.7 (figure 6). We translate scale C to place its base (graduation 1) aligned with graduation 3.1 on D. Then, we align the cursor (also known as “hairline”) on the 1.7 graduation on C. The product can be directly read on D, it’s 5.27.

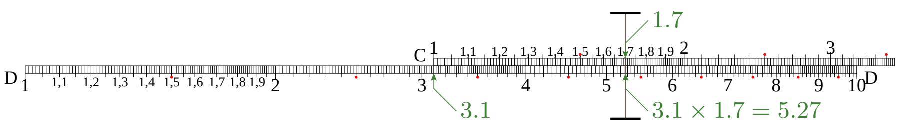

Figure 6: Multiplying 3.1 by 1.7.

3.4 Out of range product

Sometimes the product of two numbers in the range $[1..10]$ overflows 10. The logarithmic scale being graduated only up to 10, how can we proceed? In that case, we do not compute directly the product $a \times b$, but $a \times b/10$ instead, which is in the interval $[1..10]$. Dividing b by 10 is the same as translating everything to the left of a length of l_{10} (length corresponding to the distance between graduations 1 and 10). It’s a property from the logarithms: $\log(b/10) = \log(b) - \log(10) = \log(b) - 1$.

Technically, this is the same as aligning, not the 1 of scale D, but the opposite extremity (the 10), with the graduation corresponding to a on scale C. The product (divided by 10) can be read in front of the b graduation on scale D (figure 7).

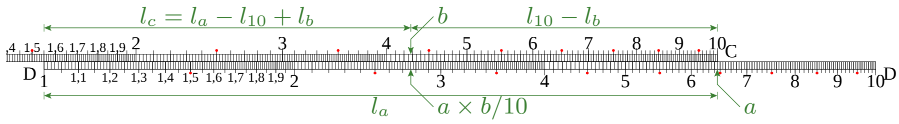

Figure 7: Out of range product.

3.5 Mantissa and exponent

When a value is not in the interval $[1..10]$, we work only with the mantissa part of the number written in scientific notation, without taking into account the exponent part.

A property of the power function intervenes, which is to transform a product of powers in a power of sum. Indeed,

$$x^a \times x^b = x^{a+b}$$

thus if $a = a_m \times 10^{a_e}$ and $b = b_m \times 10^{b_e}$, the product is $a \times b = a_m \times b_m \times 10^{a_e + b_e}$. The mantissa of the product is the product of the mantissas, calculation made with numbers in the range $[1..10]$, and the exponent of the product is the sum of the exponents. One only need to adjust the exponent of the result to the desired notation (scientific or engineering).

For example, to compute the product of 1370 (1.37×10^3) by 0.121 (1.2×10^{-1}), we compute first $1.37 \times 1.2 \approx 1.66$. Then, adding exponents ($3 + (-1) = 2$) gives us the exponent of the result, thus 1.66×10^2, i.e. 166.

4 Divisions

4.1 First method

We use the same scales as for the multiplication, but in a different way. We want to compute here $c' = a'/b'$, which can be also written as $c' \times b' = a'$. This is the same as looking for the term c' which, multiplied by a factor b', gives a'. We use the same method as for multiplying, using a for a'/b', b for b', and $c = a \times b$ for a'.

We align the b' graduation on scale C with the a' graduation on scale D. The quotient is read on scale D and corresponds to the base of scale C, i.e. the 1 graduation. (figure 8).

To the opposite of the multiplication process, where the product can overflow 10, here the quotient can underflow 1. In that case, use the 10 graduation to read the result, and divide the quotient by 10.

In order to compute the quotient 3.95 by 830, align 8.3 on scale C with 3.95 from scale D. The result is less than 1, so we need to read it on scale D, below 10 from scale C, which gives 0.475. Adjusting exponents by a method identical to the one of multiplication, we obtain $3.95/830 \approx 4.75.10^{-3}$ (figure 9).

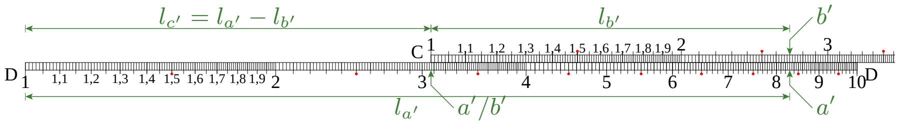

Figure 8: Division, first method.

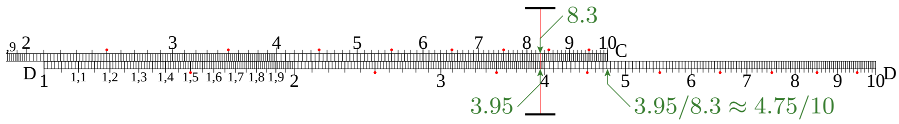

Figure 9: Dividing 3.95 by 830.

4.2 Second method

We use here a second scale CI, inverted from C (CI $\equiv$ C Inverted, in red on figure 10). The base of scale CI (graduation 1 on the right hand-side) should be aligned here with the graduation on scale D of dividend a'. Going back to the left (that is, by substracting from l_a the length l_b), the graduation on scale D aligned with b' on scale CI gives directly the result $c' = a'/b'$.

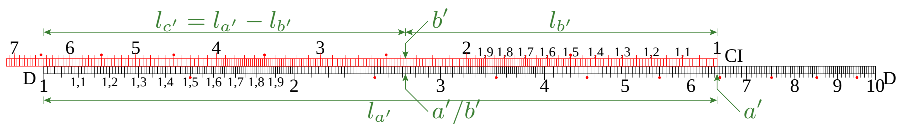

Figure 10: Dividing, second method.

If the result of the division is not in the interval $[1..10]$, align the left base (graduation 10) of CI, compute the quotient $\times 10$ and divide the result by 10.

Figure 11 show the computation of $2.14/7.65$.

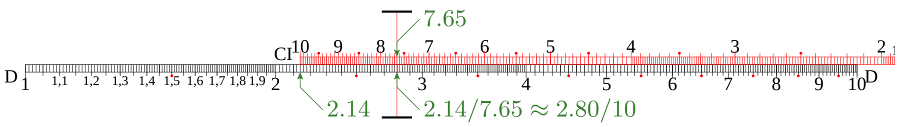

Figure 11: Dividing 2.14 by 7.65.

The main advantage of this method, apparently more complex because needing a new type of scale (CI), is clearly revealed in the section 5.

4.3 Reciprocal

The use of the two scales C and CI allows the direct computation of the reciprocal $1/a$ of a number a.

Indeed, $\log(1/a) = -\log(a)$, and $\log(10 \times 1/a) = \log(10) - \log(a)$, from where:

$$\log(10 \times 1/a) = 1 - \log(a)$$

Figure 12 explicits the computation.

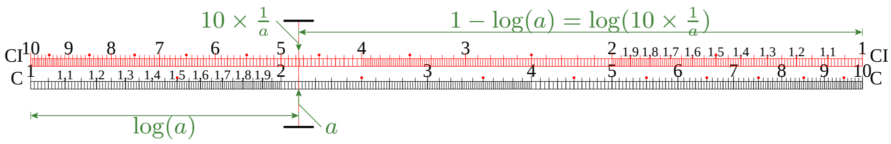

Figure 12: Computation of $1/a$.

As in multiplication, it is easy to compute the reciprocal of a number outside the interval $[1..10]$, by writing the number in scientific notation and by using the formula $1/10^b = 10^{-b}$.

5 Chaining multiplications and divisions

5.1 General method

To chain operations, like a quotient of products such as:

$$\frac{a_1 \times a_2 \times a_3}{b_1 \times b_2}$$

it is possible to compute the product $a = a_1 \times a_2 \times a_3$, keep the result a, then compute $b = b_1 \times b_2$, keep the result b, then finally divide $c = a/b$. But we need to keep two intermediate results.

There is another method, way simpler, which prevents us of writing down any intermediate results. If we use the second method to divide, the result of the division is on the D scale, ready to be re-used for the next computation stage. Thus, by alternating multiplications and divisions, we use the previous intermediate result as a base for the next computation.

The method is the following:

- Compute a_1/b_1,
- Multiply this result by a_2,
- Divide by b_2,
- Multiply then by a_3, which gives us the final result.

The first operand of stage $n + 1$ is in all cases the result of stage n. There is no intermediate result to keep.

For example we will compute

$$\frac{7.1 \times 0.51 \times 22800}{0.25 \times 61.5}$$

Only four moves of the rule are needed to obtain the result.

Compute $7.1/0.25 \approx 28.4$ (figure 13).

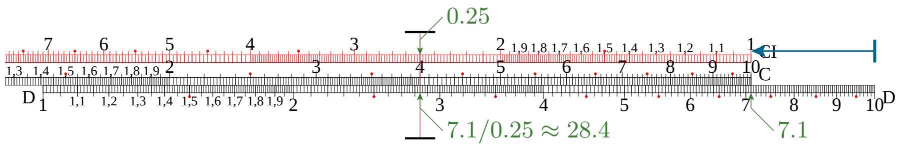

Figure 13: Chained computation, first step.

Align the 10 of the multiplication scale C on the result which was computed, that is 28.4. Multiplying by 0.51 gives 14.5 (figure 14).

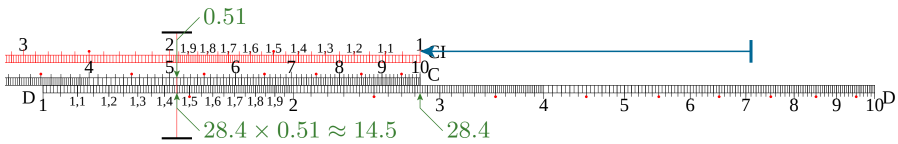

Figure 14: Chained computation, second step.

Align the 10 of the division scale CI on the result. Dividing 14.5 by 61.5 gives 0.236 (figure 15).

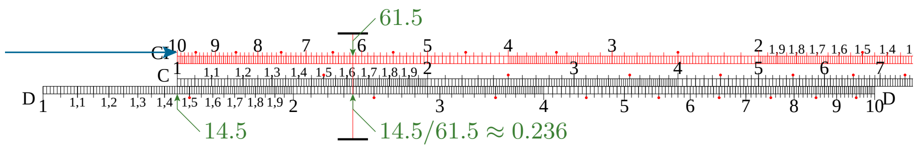

Figure 15: Chained computation, third step.

Then, align the 1 of the multiplication scale C on the result, and multiply by 22800, which gives us as a result 5380 (figure 16).

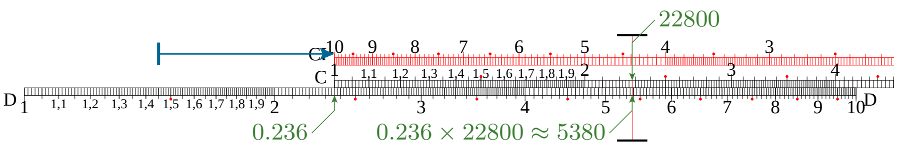

Figure 16: Chained computation, fourth and last step.

The exact result equals $5369, 678..$, which gives us an error of roughly $0, 2\%$.

5.2 Multiplying three factors

There is a quick method to multiply 3 factors $a \times b \times c$, using only one slide rule translation.

Using the CI inverted scale, we can multiply two numbers a and b by aligning them on the CI and D scale. The product $a \times b$ is then on the D scale, aligned with the base of CI scale (1). This intermediate product can then be easily reused as the first operand of a classical multiplication with the C scale, using c as the second operand. The graduation of c on C shows us directly the product $a \times b \times c$ on D (figure 17).

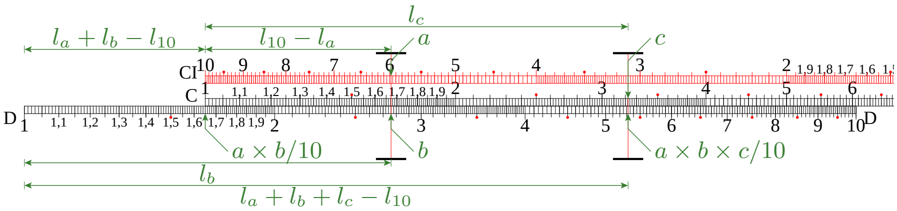

Figure 17: Multiplying 3 factors $a \times b \times c$.

6 Proportions & conversions

Proportion problems are encountered frequently, such as converting from one unit to another (inch to millimeters, knots to km/h for example), or to apply a proportion rule (how much weight 3.73 litres of a product, when 1.27 litre weight 0.965 kg). These computations are equivalent to the computation of a_1 with

$$\frac{a_1}{b_1} = \frac{a_2}{b_2}$$

knowing a_2/b_2 and b_1. This proportion rule can be written as $a_1/a_2 = b_1/b_2$, that is $\log(a_1/a_2) = \log(b_1/b_2)$. Knowing that $\log(x/y) = \log(x) - \log(y)$, the last equality can be rewritten as $\log(a_1) - \log(a_2) = \log(b_1) - \log(b_2)$. Let us pose $l_a = \log(a_1) - \log(a_2)$ and $l_b = \log(b_1) - \log(b_2)$. We can remark (figure 18) that by aligning factors a_2 et b_2 on respectively D and C scales, a_1 will be directly read on scale D aligned with b_1 on C.

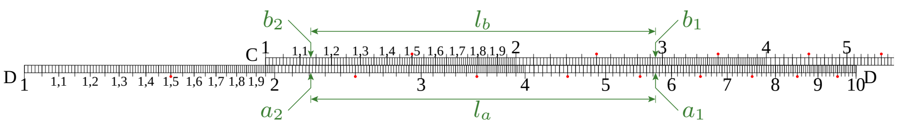

Figure 18: Proportions

In other words, the distance from b_1 to b_2 on scale C is the same as the distance from a_1 to a_2 on scale D. Once the scales setup, one can convert how many values he wants, without modifying the sliding rule ratio.

When converting out of range values, shift the C scale of a length l_{10} to the left, to bring back the 10 index in place of the 1 index, without forgetting to adjust the computed result of a factor 10.

Figure 19 gives an example of converting with a ratio of $1.74/2.44$.

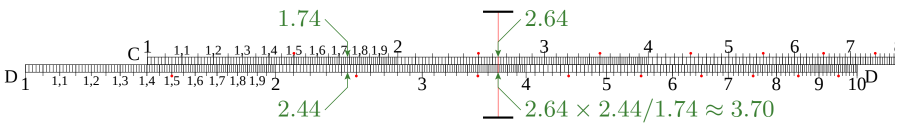

Figure 19: 3.70 is to 2.64 what 2.44 is to 1.74.

7 Squares & square roots

7.1 Squares

We use a scale, called A, graduated according to the logarithm of the square root. The length from the origin l of a graduation a is given by:

$$l = \log(\sqrt{a})$$

To compute the square of a number a, we slide the cursor over the graduation for a on the D scale. The b value aligned on scale A equals to this square: $b = a^2$. We have $l_a = l_b$, $l_a = \log(a)$ and $l_b = \log(\sqrt{b})$, so $\log(a) = \log(\sqrt{b})$, thus $b = a^2$ (figure 20).

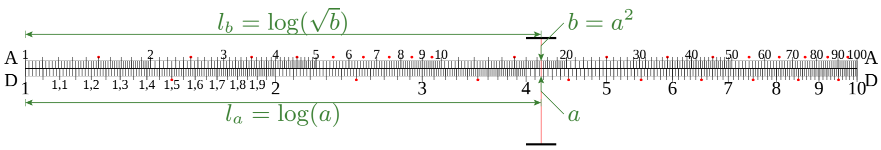

Figure 20: Square computation.

In order to compute the square of a number out of the range $[1..10]$, by writing this number on scientific form $a = a_m.10^{a_e}$, we have $a^2 = a_m^2.10^{a_e}{}^2$, or $a^2 = a_m^2.10^{2.a_e}$. The mantissa of the square is the square of the mantissa, the exponent of the square is twice the exponent. See the example of 0.388^2 on figure 21.

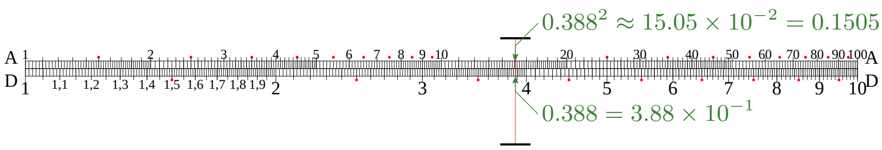

Figure 21: Square of 0.388.

7.2 Square roots

To extract the square root of a number, we could believe the opposite process would do it. The problem being that scale A has two intervals, respectively $[1..10]$ and $[10..100]$. Which one are we going to use to compute the square root of a number out of the range $[1..100]$?

We have to write the number a on which we want to compute the square, under the form $a = a_m.10^{2.a_e}$, with a_m in the interval $[1..100]$ and a_e integer. The square of a equals to:

$$\sqrt{a} = \sqrt{a_m.10^{2.a_e}}$$

thus

$$\sqrt{a} = \sqrt{a_m}.10^{a_e}$$

To compute the square root of 778312, write $778312 = 77.8312 \times 10^4$, compute $\sqrt{77.8} \approx 8.82$, we obtain then $\sqrt{778312} \approx 8.82 \times 10^2 = 882$.

8 Cubes and cube roots

The method is based on the same principle as computing squares and square roots. The scale used here, called K, is graduated upon:

$$l = \log(\sqrt[3]{a})$$

To compute the cube root, write the number a on which we want to compute the root under the form $a = a_m.10^{3.a_e}$, with a_m in the interval $[1..1000]$ and a_e integer. The remaining of the procedure is straightforward.

9 Usual calculations

9.1 Converting degree/minute

We found, on the C and D scale of most slide rules, some constants: ρ', ρ'' et $\rho_{''}$ (figure 22). This constants allows, by aligning the base of a scale on one of them, to transform angles expressed using the DMS system (Degree, Minute, Second) onto angle in radian (arc length), or the reverse.

- $\rho' = \frac{360 \times 60}{2\pi} \approx 3437,747$, converting an angle in minutes,
- $\rho'' = 60 \times \rho' \approx 206265$, converting an angle in sexagesimal seconds,
- $\rho_{''} = 100 \times \rho' \approx 636619$, converting an angle in decimal seconds.

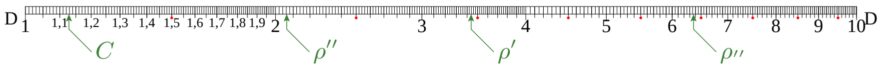

Figure 22: Usual conversion constants.

9.2 Volume of a cylinder

The constant

$$C = \sqrt{\frac{4}{\pi}} \approx 1,128379$$

allows the calculation of the volume V of a cylinder of diameter d and height h:

$$V = \frac{\pi d^2}{4} \times h$$

This equation can also be written under the form:

$$\sqrt{V} = \frac{d}{C} \times \sqrt{h}$$

Align the mark of the constant C on scale C with d on scale D. Slide the cursor on h on scale B, the volum V can then be directly read on scale A. Figure 23 visually explains the computation.

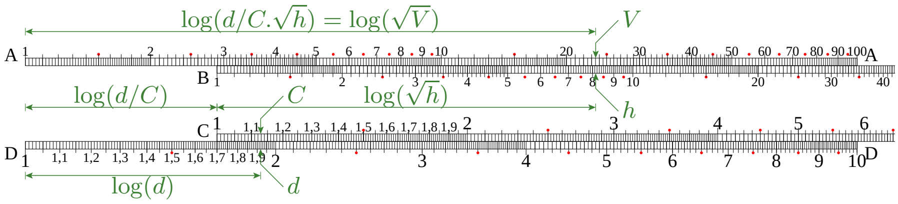

Figure 23: Cylinder's volume.

9.3 π-Shifted scales

On some slide rules we found scales CF, DF, even CIF. Theses scale are identical to respectively C, D and CI, but shifted by a π factor, on the right by a length $l_\pi = \log(\pi) \approx 0.497$. With theses scale one can perform all classical computations, but multiplied/divided with a π factor. For example, to compute $\pi \times 2.3 \times 1.7$, place the origin of scale C aligned to 2.3 on D, then read the result on DF: 12.28.

Without using the shifted scales, we can use the π constant on scales A, B, C and D to easily perform theses computations. The 3-factor multiplication method (§5.2) could be used in that case.

10 Trigonometry

Scale S (figure 24), graduated according to the law $l = \log(10.\sin(\alpha))$, allows the computation of the sine of an angle from $5.74°$ to $90°$ ($\sin(5.74°) \approx 0.1, \sin(90°) = 1$, that is, the interval of the C logarithmic scale divided by 10).

To compute the sine of an angle, directly read on the logarithmic scale the result and divide it by 10. To compute the angle for which the sine is known, proceed in the same manner but interverting the two scales.

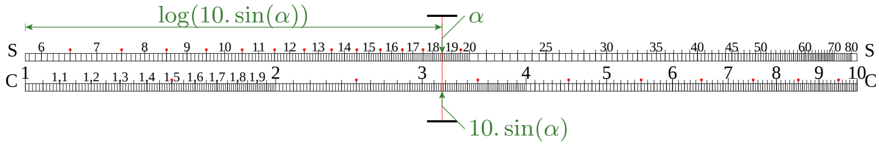

Figure 24: Sine scale from $5.7°$ to $90°$.

The computation of the tangent and its inverse is made using the same principle on a scale T, graduated upon the law $l = \log(10.\tan(\alpha))$ (figure 25).

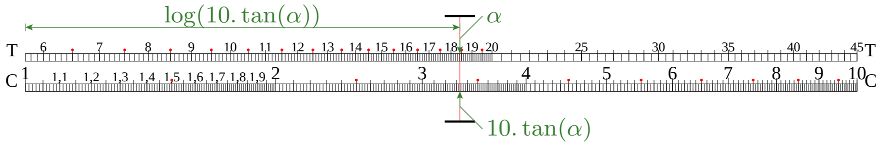

Figure 25: Tangent scale from $5.7°$ to $45°$.

To compute the sine or the tangent of an angle in the range $0.573°$ and $5.7°$ ($\sin(0.573°) \approx \tan(0.573°) \approx 0.01$), we use in both cases a unique scale called ST (figure 26), graduated according to the law $l = \log(100.(\sin(\alpha) + \tan(\alpha))/2)$. We can use one rule for both because for small angles, sine and tangent are nearly the same. The error, ϵ, between the sine and the tangent for a given angle α, is:

$$\epsilon = 2.\frac{\tan(\alpha) - \sin(\alpha)}{\tan(\alpha) + \sin(\alpha)}$$

which gives 0.496% for $\alpha = 5.7°$; and only 0.005% for $\alpha = 0.57°$. The error is in the same order of magnitude as the reading error, and greatly less in many cases, so is negligible.

Lastly, for angles lower than $0,573°$, the sine and the tangent of an angle can be directely approximated by the arc length (the angle in radians). The error between the arc length and the sine for an angle α is:

$$\epsilon = \frac{\alpha - \sin(\alpha)}{\alpha}$$

equals to 0.0016% for $\alpha = 0.57°$, negligible in all the cases compared to the general precision.

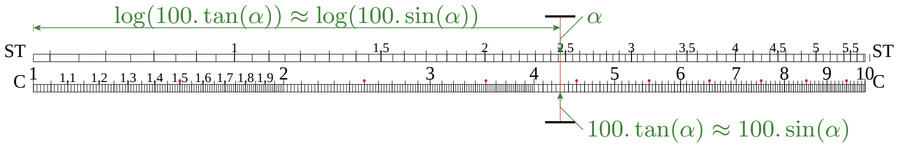

Figure 26: Sine and tangent scale from $0.57°$ to $5.7°$.

The computation of the cosine, sine or tangent of an angle outside the interval $[0..90°]$ is directly deduced from the sine or tangent of an angle from this interval by the usual trigonometric formulae:

$$\cos(\alpha) = \sin(\frac{\pi}{2} + \alpha)$$

$$\sin(-\alpha) = \sin(\pi + \alpha) = -\sin(\alpha)$$

$$\tan(\frac{\pi}{2} + \alpha) = \frac{-1}{\tan(\alpha)}$$

$$\tan(-\alpha) = \tan(\pi - \alpha) = -\tan(\alpha)$$

11 Logarithms, exponential, powers

11.1 Decimal logarithms

The computation of a logarithm is straightforward when one has a logarithmic scale. The logarithm of a number a on the D scale is equals to the length l_a: the scale L is thus simply linear (figure 27).

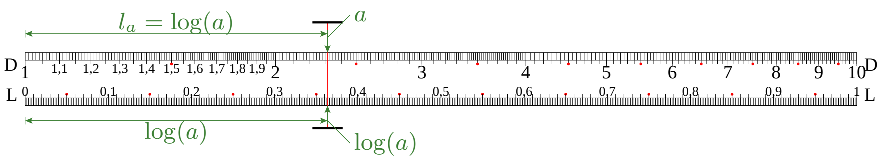

Figure 27: Logarithms.

To compute logarithms of numbers greater than 10 or smaller than 1, we write a as: $a = a_m.10^{a_e}$, then $\log(a) = \log(a_m.10^{a_e}) = \log(a_m) + \log(10^{a_e})$, knowing that $\log(10^x) = x$, we obtain:

$$\log(a) = \log(a_m) + a_e$$

If we write the number under a scientific form, the logarithm is equals to the exponent a_e added to the logarithm of the mantissa a_m.

11.2 Decimal powers

To compute 10^a, we have to split the integer part a_e (also called characteristic) from the decimal part a_f (also called mantissa) of a: $a = a_e + a_f$.

We can remark that $10^a = 10^{a_e + a_f}$ can also be written as $10^a = 10^{a_e}.10^{a_f}$. It is easy to compute 10^{a_f} using the linear scale L and the logarithmic scale D (figure 28), it is the mantissa of the result written in scientific form. 10^{a_e} is straightforward to compute and is the exponent part of the same result.

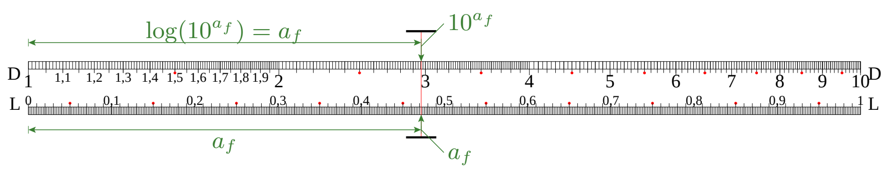

Figure 28: Power of 10.

For example: $10^{2.472} = 10^2 \times 10^{0.472}$. Using the cursor with 0.472 on scale L we found $10^{0.472} \approx 2.965$ on scale D. Thus, $10^{2.472} \approx 2.965 \times 10^2$, which is approximatively 296.5.

11.3 Powers

The objective here is to calculate $c = a^b$.

First method We compute the logarithm of the two members of the expression: $\log(c) = \log(a^b)$. We have $\log(a^b) = b.\log(a)$, so:

$$c = 10^{b.\log(a)}$$

The procedure is thus the following:

- Compute the logarithm of a (see §11.1),
- Multiply this logarithm by b,
- Compute the power of 10 of the last result (see §11.2).

This method, though indirect, is using only scales C, D and L, and works with all numbers a and b (Except for the necessary adjustments).

Second method We use a scale LL (logarithm of the logarithm, or log-log), graduated according to the law $l = \log(k.\ln(a))$. Indeed, $\log(k.\ln(a^b)) = \log(k.b.\ln(a))$, so

$$\log(k.\ln(a^b)) = \log(b) + \log(k.\ln(a))$$

Number a on scale LL gives us a length l_a, on which we add the distance l_b given by b on log scale C. The value c on scale LL aligned with b on scale C is the result of a to the power b: a^b (figure 29).

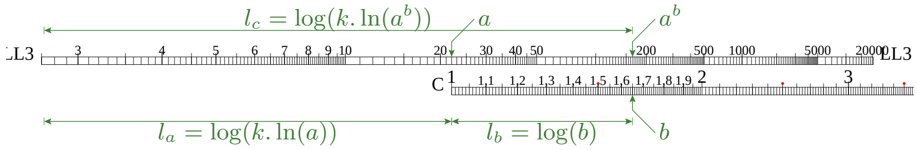

Figure 29: Calculating a^b.

LL scales are not relatives but absolutes: they are graduated with the logarithm of a logarithm, thus the rule $\log(10.a) = 1 + \log(a)$ cannot be applied. We then have to use several scales according to the interval we should use. Usual LL scales are:

- LL3: $l_3 = \log(\ln(a))$ (interval $[e..e^{10}]$),
- LL2: $l_2 = \log(10.\ln(a))$ (interval $[e^{0,1}..e]$),
- LL1: $l_1 = \log(100.\ln(a))$ (interval $[e^{0,01}..e^{0,1}]$).

The end of scale LL1 corresponds to the beginning of scale LL2, the end of LL2 to the beginning of LL3 (e being the base of natural logarithms).

To calculate $1.147^{4.32}$ (figure 30), we align the origin of scale C on 1.147 of scale LL2. The result, ≈ 1.808, can be read using the cursor on LL2 aligned with 4.32 on scale C.

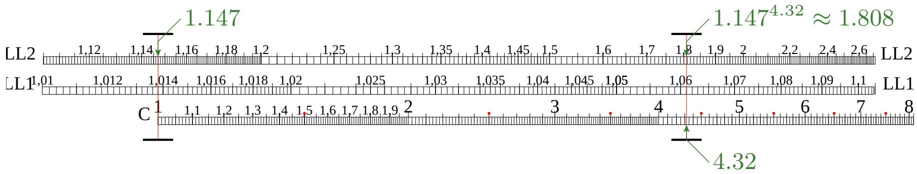

Figure 30: Calculating $1.147^{4.32}$.

11.4 Natural logarithms, exponential

We know that $\ln(a) = \ln(10).\log(a)$, so:

$$\ln(a) \approx 2.303.\log(a)$$

This is equivalent to compute a decimal logarithm which can then be multiplied by a constant.

To compute an exponential without using a LL scale, we use the first method of computing a power (§ 11.3). $\log(e^a) = a.\log(e)$, we should then remind us the constant $\log(e) \approx 0.434$. We only need to multiply a with this constant, 0.434, and read the result on scale L, with all the necessary adjustments.

11.5 Squares

To compute $c = \sqrt[b]{a}$, we can remark that $\sqrt[b]{a} = a^{1/b}$. The calculation is the same as a power, with the exponent being the reciprocal of the square base.